University of Mannheim
School of Social Sciences
Mathematics for Political Scientists, Fall 2022
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Solutions Linear Algebra I

1. Consider the following matrices and vectors.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 8 & 3 \\ 0 & -1 & 6 \end{pmatrix};\ \mathbf{B} = \begin{pmatrix} -3 & 2 & 4 \\ 2 & 3 & 4 \\ 2 & -4 & 0 \end{pmatrix};\ \mathbf{c} = \begin{pmatrix} 4 & -3 & 2 \end{pmatrix};\ \mathbf{d} = \begin{pmatrix} 3 & 8 \end{pmatrix};$$

$$\mathbf{e} = \begin{pmatrix} 2 & 6 & 9 \end{pmatrix};\ \mathbf{F} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix};\ \mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ \mathbf{H} = \begin{pmatrix} 5 & 6 & 1 \\ -2 & 7 & 8 \end{pmatrix};$$

$$\mathbf{K} = \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix}$$

Do the calculations if possible.

(a) $\mathbf{M}_1 = \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 13 & -9 & 16 \\ 16 & 16 & 40 \\ 10 & -27 & -4 \end{pmatrix}$

(b) $\mathbf{M}_2 = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 & 1 & 1 \\ 0 & 5 & -1 \\ -2 & 3 & 6 \end{pmatrix}$

(c) $\mathbf{M}_3 = \mathbf{B} \cdot \mathbf{F} = n.d.$
Not possible since ncol($\mathbf{B}$)$\neq$nrow($\mathbf{F}$).

(d) $\mathbf{M}_4 = \mathbf{A} \cdot \mathbf{c} = n.d.$
Not possible since ncol($\mathbf{A}$)$\neq$nrow($\mathbf{c}$)

(e) $\mathbf{M}_5 = \mathbf{c} \cdot \mathbf{A} = \begin{pmatrix} -2 & -14 & 23 \end{pmatrix}$

(f) $\mathbf{m}_6 = \mathbf{d} \cdot \mathbf{c} = n.d.$
Not possible since the vectors have different dimension.

(g) $\mathbf{m}_7 = 2\mathbf{c} \cdot 3\mathbf{e} = 48.$

(h) $\mathbf{M}_8 = \mathbf{B} \cdot \mathbf{G} = \mathbf{B}$

(i) $\mathbf{M}_9 = \mathbf{A} \cdot \mathbf{H} = n.d.$
Not possible since ncol($\mathbf{A}$)$\neq$ nrow($\mathbf{H}$)

(j) $\mathbf{M}_{10} = \mathbf{H}' \cdot \mathbf{F} = \begin{pmatrix} 13 & -4 \\ 25 & 14 \\ 11 & 16 \end{pmatrix}$

2. What is the dimension of the following matrices?

   (a) $\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{H}'$ $\quad 3 \times 2$

   (b) $\mathbf{c} + \mathbf{e} \cdot \mathbf{H}'$ $\quad$ not possible, since ncol($\mathbf{c}$) = 3 and ncol($\mathbf{e} \cdot \mathbf{H}'$) = 2

   (c) $\mathbf{F} \cdot \mathbf{K}$ $\quad 2 \times n$

3. Specify whether the following matrices are square, zero, identity, diagonal, or upper/lower triangular matrices and give their dimension as well as their rank.

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 & 0 & 8 \\ 0 & 1 & -2 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 7 & 0 \\ 1 & -3 & 9 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 8 & 0 \\ 0 & -5 & 0 \end{pmatrix}$$

	A	**B**	**C**	**D**	**E**
dimension	(3×2)	(2×2)	(2×3)	(3×3)	(3×3)
rank	0	2	2	3	2
square matrix		x		x	x
zero matrix	x				
identity matrix		x			
diagonal matrix		x			
upper triangular matrix	(x)	x	(x)		
lower triangular matrix	(x)	x			x

4. Is the equation $(\mathbf{F} + \mathbf{G})^2 = \mathbf{F}^2 + 2 \cdot \mathbf{F} \cdot \mathbf{G} + \mathbf{G}^2$ true for any square matrices of the same order?

   $(\mathbf{F} + \mathbf{G})^2 = (\mathbf{F} + \mathbf{G}) \cdot (\mathbf{F} + \mathbf{G}) = \mathbf{F}^2 + \mathbf{F} \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{F} + \mathbf{G}^2$
   No, this is only the case if $\mathbf{F} \cdot \mathbf{G} = \mathbf{G} \cdot \mathbf{F}$.

5. Find all 2×2 matrices $\mathbf{A}$ such that $\mathbf{A}^2$ is the matrix obtained from $\mathbf{A}$ by squaring each entry.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aa + bc & ab + bd \\ ac + cd & bc + dd \end{pmatrix} \equiv \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$$

   From the identity we derive the following system of equations.

$$aa + bc = aa \quad (1)$$
$$ab + bd = bb \quad (2)$$
$$ac + cd = cc \quad (3)$$
$$bc + dd = dd \quad (4)$$

From (1) and (4) we know that $bc = 0$. The first and easy solution, thus, is all matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

A little bit more subtle are the other two possible solutions

$$\begin{pmatrix} a & 0 \\ a+d & d \end{pmatrix} \text{ and } \begin{pmatrix} a & a+d \\ 0 & d \end{pmatrix}.$$

For the proof, note that either b or c has to be equal to zero. For symmetry we focus on the case where $b = 0$. Let $c = a + d$, then equation (3)

$$\begin{aligned} ac + cd &= cc \\ a(a+d) + d(a+d) &= (a+d)(a+d) \\ a^2 + 2ad + d^2 &= (a+d)^2. \end{aligned}$$