University of Mannheim
School of Social Sciences
Mathematics for Political Scientists, Fall 2022
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Solutions Analysis I

1. Solve the following equations.

(a)

$$\begin{aligned} x^2 - 6x + 8 &= 0 \\ x_{1|2} &= \frac{6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 8}}{2 \cdot 1} \\ x_1 &= 4 \\ x_2 &= 2 \end{aligned}$$

(b)

$$\begin{aligned} (3x-1)^2 - (5x-3)^2 &= -(4x-2)^2 \\ 9x^2 - 6x + 1 - 25x^2 + 30x - 9 &= -16x^2 + 16x - 4 \\ 8x &= 4 \\ x &= 0.5 \end{aligned}$$

(c)

$$\begin{aligned} \sqrt{x^2 - 9} &= 9 - x \\ x^2 - 9 &= (9-x)^2 \\ x^2 - 9 &= x^2 - 18x + 81 \\ 18x &= 90 \\ x &= 5 \end{aligned}$$

(d)

$$\begin{aligned} \log_x(2x+8) &= 2 \\ 2x + 8 &= x^2 \\ x_{1|2} &= 1 \pm 3 \end{aligned}$$

(e)

$$\begin{aligned} e^{2x-5}+1 &= 4 \\ e^{2x-5} &= 3 \\ 2x-5 &= \ln 3 \\ x &= 0.5\ln 3 + 2.5 \end{aligned}$$

(f)

$$\begin{aligned} \log_2 \frac{2}{x} &= 3+\log_2 x \\ \frac{2}{x} &= 2^{3+\log_2 x} \\ \frac{2}{x} &= 2^3 \cdot 2^{\log_2 x} \\ \frac{2}{x} &= 8\cdot x \\ x^2 &= 0.25 \\ x &= 0.5 \end{aligned}$$

(g)

$$\begin{aligned} (27)^{2x+1} &= \frac{1}{3} \\ 2x+1 &= \log_{27}\frac{1}{3} \\ 2x+1 &= -\frac{1}{3} \\ x &= -\frac{2}{3} \end{aligned}$$

2. Simplify the following expressions.

(a)

$$\begin{aligned} \frac{4^2\cdot 6^2}{3^3\cdot 2^3} &= \frac{(4\cdot 6)^2}{(3\cdot 2)^3} \\ &= \frac{24^2}{6^3} \\ &= \frac{1}{6}\cdot\frac{24^2}{6^2} \\ &= \frac{1}{6}4^2 \end{aligned}$$

(b)

$$\begin{aligned} \frac{(x+1)^3(x+1)^{-2}}{(x+1)^2(x+1)^{-3}} &= \frac{(x+1)^3\cdot(x+1)^3}{(x+1)^2\cdot(x+1)^2} \\ &= \frac{(x+1)^6}{(x+1)^4} \\ &= (x+1)^2 \end{aligned}$$

(c)

$$(-3xy^2)^3 = -27x^3y^6$$

(d)

$$\begin{aligned}
\frac{\frac{(x^2)^3}{x^4}}{\left(\frac{x^3}{(x^3)^2}\right)^{-2}} &= \frac{\frac{x^{2\cdot 3}}{x^4}}{\frac{x^{3\cdot(-2)}}{x^{3\cdot 2\cdot(-2)}}} \\
&= \frac{\frac{x^6}{x^4}}{\frac{x^{12}}{x^6}} \\
&= \frac{x^6}{x^4}\cdot\frac{x^6}{x^{12}} \\
&= \frac{1}{x^4}
\end{aligned}$$

(e)

$$\begin{aligned}
((2x+1)(2x-1))\,(4x^2+1) &= (4x^2-2x+2x-1)(4x^2+1) \\
&= (4x^2-1)(4x^2+1) \\
&= 16x^4-1
\end{aligned}$$

(f)

$$\begin{aligned}
\frac{6x^5+4x^3-1}{2x^2} &= \frac{6x^5+4x^3}{2x^2}-\frac{1}{2x^2} \\
&= 3x^3+2x-\frac{1}{2x^2}
\end{aligned}$$

(g)

$$\begin{aligned}
\frac{1+4x^2+6x}{2x-1} &= \frac{(4x^2+1)+6x}{2x-1} \\
&= \frac{(2x+1)(2x-1)+2+6x}{2x-1} \\
&= 2x+1+\frac{6x+2}{2x-1} \\
&= 2x+1+\frac{(6x-3)+5}{2x-1} \\
&= 2x+1+3+\frac{5}{2x-1} \\
&= 2x+4+\frac{5}{2x-1}
\end{aligned}$$

(h)

$$\begin{aligned}\frac{x^2-5x+4}{x^2+2x-3}-\frac{x^2+2x}{x^2+5x+6} &= \frac{(x-1)(x-4)}{(x-1)(x+3)}-\frac{x(x+2)}{(x+2)(x+3)}\\ &= \frac{x-4}{x+3}-\frac{x}{x+3}\\ &= \frac{x-4-x}{x+3}\\ &= -\frac{4}{x+3}\end{aligned}$$

3. Show that:

(a) $\sum_{i=1}^{N}(x_i-\mu_x)^2=\sum_{i=1}^{N}x_i^2-N\mu_x^2$. Hint: Note that $\mu_x=\frac{1}{N}\sum_{i=1}^{N}x_i$.

$$\begin{aligned}\sum_{i=1}^{N}(x_i-\mu_x)^2 &= \sum_{i=1}^{N}(x_i^2-2\mu_x x_i+\mu_x^2)\\ &= \sum_{i=1}^{N}x_i^2-2\mu_x\sum_{i=1}^{N}x_i+\sum_{i=1}^{N}\mu_x^2\\ &= \sum_{i=1}^{N}x_i^2-2\mu_x N\mu_x+N\mu_x^2\\ &= \sum_{i=1}^{N}x_i^2-N\mu_x^2.\end{aligned}$$

(b) $\sum_{i=1}^{n}(a_{i+1}-a_i)=a_{n+1}-a_1$.

$$\begin{aligned}\sum_{i=1}^{N}(a_{i+1}-a_i) &= \sum_{i=1}^{N}a_{i+1}-\sum_{i=1}^{N}a_i\\ &= \sum_{i=1}^{N}a_i+a_{N+1}-a_1-\sum_{i=1}^{N}a_i\\ &= a_{N+1}-a_1.\end{aligned}$$

4. Show that $a^3-b^3=(a-b)(a^2+ab+b^2)$

$$\begin{aligned}a^3-b^3 &= a(a^2-b^2)+ab^2-b^3\\ &= a(a+b)(a-b)+b^2(a-b)\\ &= (a-b)(a^2+ab+b^2)\end{aligned}$$

5. Differentiate the following functions with respect to x.

(a) $f(x)=7x^3-2x^2-5x+1$
$f'(x)=21x^2-4x-5$

(b) $f(x) = 0.7x^{-4} + 1.3 - 3.1x^3$
$f'(x) = -2.8x^{-5} - 9.3x^2$

(c) $f(x) = \dfrac{3x^2+1}{2x}$
$f'(x) = \dfrac{3}{2} - \dfrac{1}{2x^2}$

(d) $f(x) = \sqrt{4x+9}$
$f'(x) = \dfrac{2}{\sqrt{4x+9}}$

(e) $f(x) = \dfrac{x^{\frac{1}{3}} - 2}{(x^5-2)^3}$
$f'(x) = \dfrac{\frac{1}{3}x^{-\frac{2}{3}} \cdot (x^5-2)^3 - (x^{\frac{1}{3}} - 2) \cdot 3(x^5-2)^2 \cdot 5x^4}{(x^5-2)^6}$

(f) $f(x) = \ln\left(\dfrac{x^2}{x^4+1}\right)$
$f'(x) = \dfrac{x^4+1}{x^2} \cdot \dfrac{2x(x^4+1) - x^2 \cdot 4x^3}{(x^4+1)^2} = \dfrac{2}{x} - \dfrac{4x^3}{x^4+1}$

(g) $f(x) = e^{x^3+x}$
$f'(x) = e^{x^3+x} \cdot (3x^2+1)$

(h) $f(x) = \dfrac{1}{e^x + e^{-x}}$
$f'(x) = -1(e^x + e^{-x})^{-2} \cdot (e^x - e^{-x}) = -\dfrac{e^x - e^{-x}}{(e^x+e^{-x})^2}$

6. Find the all first and second (mixed) partial derivatives of the following functions.

(a) $f(x,y) = \ln x \cdot y^2$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{y^2}{x} \\
\frac{\partial^2 f}{\partial x^2} &= -\frac{y^2}{x^2} \\
\frac{\partial f}{\partial y} &= 2y \ln x \\
\frac{\partial^2 f}{\partial y^2} &= 2 \ln x \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{2y}{x}
\end{aligned}$$

(b) $f(x,y) = \sqrt{2x - y}$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= (2x-y)^{-\frac{1}{2}} \\
\frac{\partial^2 f}{\partial x^2} &= -(2x-y)^{-\frac{3}{2}} \\
\frac{\partial f}{\partial y} &= -\frac{1}{2}(2x-y)^{-\frac{1}{2}} \\
\frac{\partial^2 f}{\partial y^2} &= -\frac{1}{4}(2x-y)^{-\frac{3}{2}} \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{1}{2}(2x-y)^{-\frac{3}{2}}
\end{aligned}$$

(c) $f(x,y) = (x+4y)(e^{-2x}+e^{-3y})$

$$
\begin{aligned}
f(x,y) &= (x+4y)(e^{-2x}+e^{-3y}) \\
&= xe^{-2x} + xe^{-3y} + 4ye^{-2x} + 4ye^{-3y} \\
\frac{\partial f}{\partial x} &= e^{-2x} + x \cdot (-2) \cdot e^{-2x} + e^{-3y} - 8ye^{-2x} \\
&= (1-2x-8y)e^{-2x} + e^{-3y} \\
\frac{\partial^2 f}{\partial x^2} &= -2(1-2x-8y)e^{-2x} - 2e^{-2x} \\
&= 4(-1+x+4y)e^{-2x} \\
\frac{\partial f}{\partial y} &= -yxe^{-3y} + 4e^{-2x} + 4e^{-3y} - 12ye^{-3y} \\
&= (4-xy-12y)e^{-3y} + 4e^{-2x} \\
\frac{\partial^2 f}{\partial y^2} &= -3(4-xy-12)e^{-3y} - 12e^{-3y} \\
&= 3(-8+xy+12)e^{-3y} \\
\frac{\partial^2 f}{\partial x \partial y} &= -8e^{-2x} - 3e^{-3y}
\end{aligned}
$$

7. For what value of a is the following function continuous for all x? Is it also differentiable for all x for this value of a?

$$
f(x) = \begin{cases} ax-1 & \text{if } x \leq 1 \\ 3x^2+1 & \text{if } x > 1 \end{cases}
$$

Take the limit of the (sub)function $3x^2+1$ at $x_0 = 1$

$$
\begin{aligned}
f(x_0+h) &= \lim_{h \to 0} 3(x_0+h)^2 + 1 \\
f(1+h) &= 3 \lim_{h \to 0} (1+h)^2 + 1 \\
&= 4
\end{aligned}
$$

To have continuity, both (sub)functions have to have the same output in the limit. Hence, we equate 4 with the $a-1$. We receive $a = 5$.

To check for differentiability, we compute the derivative of $g(x) = 5x - 1$, which is $g'(x) = 5$ and $g'(1) = 5$. Since 1 is not in the domain of $h(x) = 3x^2+1$, we have to take the limit:

$$
\begin{aligned}
h'(x_0) &= \lim_{h\to 0} \frac{(3(x_0+h)^2+1)-(3x_0^2+1)}{h} \\
h'(1) &= \lim_{h\to 0} \frac{(3(1+h)^2+1)-(3+1)}{h} \\
&= \lim_{h\to 0} \frac{3(1+2h+h^2)+1-3-1}{h} \\
&= \lim_{h\to 0} \frac{6h+3h^2}{h} \\
&= \lim_{h\to 0} 6+3h \\
&= 6
\end{aligned}
$$

Since the derivatives at $x_0 = 1$ are not the same, the function can not be differentiable at $x_0 = 1$.