University of Mannheim
School of Social Sciences
Mathematics for Political Scientists, Fall 2022
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Solutions Analysis II

1. Suppose the function f is defined for all $x \in [-1.5, 2.5]$ by $f(x) = x^5 - 5x^3$.

   (a) Determine for which values of x the value of the function is equal to zero.

   $$\begin{aligned} x^5 - 5x^3 &= 0 \\ x^5 &= 5x^3 \\ x^2 &= 5 \\ x &= \pm\sqrt{5} \end{aligned}$$

   From the second equation we see that $x = 0$ is a possible solution. For $x = \pm\sqrt{5}$ we have to check whether these points are in our domain. This is true for $x = \sqrt{5}$, but not for $x = -\sqrt{5}$. Thus, the function has two roots.

   (b) Calculate $f'(x)$ and find the extreme points of f. What is the maximum/the minimum of the function.
   $f'(x) = 5x^4 - 15x^2$. The FOC gives us.

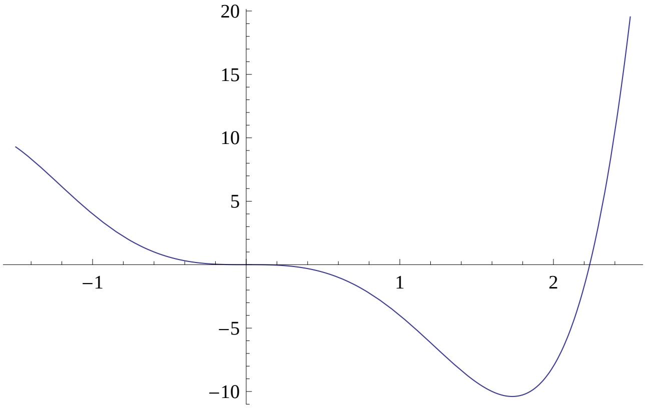
   $$\begin{aligned} 5x^4 - 15x^2 &= 0 \\ 5x^4 &= 15x^2 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

   When checking for the domain, we find that $x = 0$ and $x = \sqrt{3}$ serve as possible extreme points. Now we need to check the SOC.

   $$\begin{aligned} f''(x) &= 20x^3 - 30x \\ f''(x = 0) &= 0 \\ f''(x = \sqrt{3}) &= 30\sqrt{3} > 0 \end{aligned}$$

   At $x = 0$ we have a saddle point. At $x = \sqrt{3}$ there is a minimum.
   Are there any other minima/maxima? Yes, of course. We have to consider the boundaries of our domain. Both at $x = -1.5$ and $x = 2.5$ we have additional maxima.
   The overall maximum of the function is attained at $x = 2.5$ with $f(x) \approx 19.5$.
   The overall minimum is $x = \sqrt{3}$ with $f(x) \approx -10.4$.

(c) Does the function have inflection points?
Yes, it does. We already found the first inflection point, which also happens to be a saddle point.
We find the additional inflection points by setting $f''(x) = 0$. This gives $x = \pm\sqrt{1.5}$.



2. Which of the following functions of x are convex? Which are concave?

(a) $f(x) = (2x-1)^6$
$f'(x) = 6(2x-1)^5 \cdot 2$
$f''(x) = 5 \cdot 12(2x-1)^4 \cdot 2 \geq 0 \Longrightarrow$ convex

(b) $f(x) = 5x + 7$
The function is both convex and concave since the sets of points above and below the function are convex.

(c) $f(x) = x^5$
$f'(x) = 5x^4$
$f''(x) = 20x^3$
The function as a whole is neither convex nor concave (but we can specify this for parts of the function).

(d) $f(x) = \sqrt{1+x^2}$
$f'(x) = x(1+x^2)^{-\frac{1}{2}}$
$f''(x) = (1+x^2)^{-\frac{1}{2}} + x^2(1+x^2)^{-\frac{3}{2}} > 0 \Longrightarrow$ strictly convex

(e) $f(x) = x^5$ for $x \geq 0$
$f''(x) = 20x^3 \geq 0 \ \forall \ x \geq 0 \Longrightarrow$ convex

(f) $f(x) = 5x^2 - x^4$ for $x \geq 1$
$f'(x) = 10x - 4x^3$
$f''(x) = 10 - 12x^2 < 0 \ \forall \ x \geq 1 \Longrightarrow$ strictly concave

3. Appeasement Problem (Ashworth and Bueno de Mesquita, 2006). For full text see exercise set.

(a) Take the derivative with respect to x, set up the FOC, and solve for x.

$$\begin{aligned} 1 - 2x - q &= 0 \\ \underset{2}{x^*}(q) &= \frac{1-q}{2} \end{aligned}$$

$x^*(q)$ represents state S's optimal choice of appeasement as a function of S's perceived military strength.

(b) We can find comparative statics by examining how this equilibrium offer ($x^*(q)$) changes when q changes. Differentiating $x^*(q)$ with respect to q yields:

$$\frac{\partial x^*(q}{\partial q} = -\frac{1}{2} < 0$$

Not surprisingly, the optimal offer is decreasing in q. The stronger S is militarily, the less willing S is to appease D.

4. A government has to decide about the allocation of its budget. Let x denote the share of the budget used for military and y the share of the budget used for social expenditures. The government has to use of all its budget and has the following utility function:

$$u(x, y) = e^{2x} + e^{2y}$$

Solve the government's optimization problem.

The governments optimization problem is:

$$\max_{x,y} e^{2x} + e^{2y} \text{ s.t.}$$
$$x + y = 1$$

Setting up the Langrangian yields:

$$\mathcal{L} = e^{2x} + e^{2y} - \lambda(x + y - 1)$$

Taking the partial derivatives with respect to x, y together with the budget constraint gives:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 2e^{2x} - \lambda \\ \frac{\partial \mathcal{L}}{\partial y} &= 2e^{2y} - \lambda \\ x + y &= 1 \end{aligned}$$

Setting the first and the second equation equal yields:

$$\begin{aligned} 2e^{2x} - \lambda &= 2e^{2y} - \lambda \\ x &= y \end{aligned}$$

Together with the budget constraint we know that, $x = y = \frac{1}{2}$.

5. Consider the function $f(x) = (x^2 + 2x)e^{-x}$.

(a) Determine for which values of x the value of the function is equal to zero. We have to set $(x^2 + 2x)e^{-x} = 0$. We know that $e^{-x} > 0 \; \forall \; x \in \mathbb{R}$. Thus,

$$\begin{aligned} x^2 + 2x &= 0 \\ x &= \frac{-2 \pm \sqrt{4 - 0}}{2} = -1 \pm 1 \end{aligned}$$

The roots of the function are $x = -2$ and $x = 0$.

(b) Calculate f'(x) and find the extreme points of f. What is the maximum/the minimum of the function?

$$\begin{aligned}
f(x) &= (x^2+2x)e^{-x} \\
f'(x) &= -(x^2+2x)e^{-x}+(2x+2)e^{-x} \\
f''(x) &= (x^2+2x)e^{-x}-2(2x+2)e^{-x}+2e^{-x}
\end{aligned}$$

We take the FOC $f'(x)=0$ to look for stationary points.

$$\begin{aligned}
-(x^2+2x)e^{-x}+(2x+2)e^{-x} &= 0 \\
-(x^2+2x)+(2x+2) &= 0 \\
-x^2+2 &= 0 \\
x &= \pm\sqrt{2}
\end{aligned}$$

We have stationary points at $x=\pm\sqrt{2}$. We now have to check the SOC.

$$\begin{aligned}
f''(x) &= (x^2+2x)e^{-x}-2(2x+2)e^{-x}+2e^{-x} \\
&= (x^2+2x-2x-2-2x-2+2)e^{-x} \\
&= (x^2-2x-2)e^{-x}
\end{aligned}$$

Again, we know that $e^{-x}>0\ \forall\ x\in\mathbb{R}$, so that we only have to consider the polynomial. For $x=-\sqrt{2}$, the polynomial $(2+2\sqrt{2}-2)>0\implies$ local minimum.
For $x=\sqrt{2}$, the polynomial $(2-2\sqrt{2}-2)<0\implies$ local maximum.
As the domain is not limited, we have to check for the limit of $f(x)$ for $x\longrightarrow\pm\infty$ in order to specify whether the local extreme points are also global.

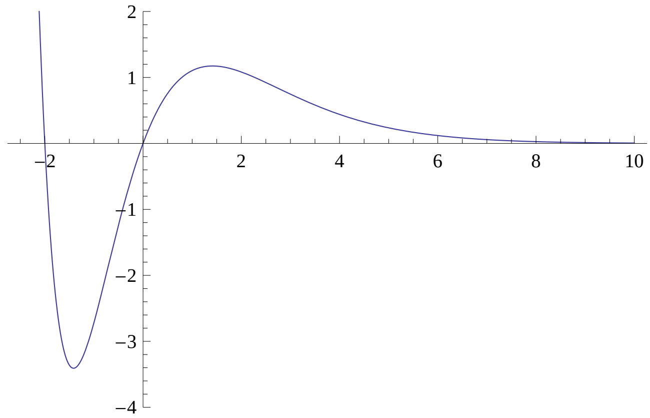
$$\begin{aligned}
\lim_{x\to-\infty}(x^2+2x)e^{-x} &\approx e^{-x}=\infty \\
\lim_{x\to\infty}(x^2+2x)e^{-x} &\approx e^{-x}=0
\end{aligned}$$

Therefore, the function does not have a global maximum. However, it has a global minimum since $f(-\sqrt{2})<\lim_{x\to\infty}f(x)$.

(c) Does the function have inflection points?
Yes, the function does have inflection points.

$$\begin{aligned}
f''(x) &= 0 \\
(x^2+2x)e^{-x}-2(2x+2)e^{-x}+2e^{-x} &= 0 \\
(x^2+2x)-2(2x+2)+2 &= 0 \\
x^2-2x-2 &= 0 \\
x &= \frac{2\pm\sqrt{4+8}}{2} \\
x &= 1\pm\sqrt{3}
\end{aligned}$$

(d) Sketch the function and specify whether it is convex/concave (in sections).



The function is neither convex nor concave as a whole.
For concavity/convexity in parts of the function the inflection points are crucial.
We can see from the graph that the function is convex for all
$x \in (-\infty, 1-\sqrt{3}]$ and $x \in [1+\sqrt{3}, \infty)$.
It is concave for all $x \in [1-\sqrt{3}, 1+\sqrt{3}]$.

6. Derivate the indefinite integrals:

(a) $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$

(b) $\int e^{-4t} dt = -\dfrac{1}{4e^{4t}} + C$

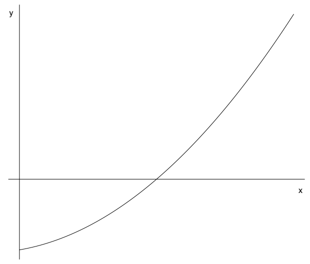
(c) $\int x\sqrt{x} dx = \frac{2}{5} x^{\frac{5}{2}} + C$

(d) $\int \dfrac{1}{x} = \ln x dx + C$

(e) $\int (2x^2 + x - 3) dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - 3x + C$

(f) $\int \dfrac{(x^4+1)^2}{x^3} dx = \frac{1}{6}x^6 + x^2 - \frac{1}{2x^2} + C$

7. Calculate $\int_0^2 (2x^2 + x - 3)dx$. Hint: Make a sketch of the function before.



At $x = 1$, the curve crosses the horizontal axis. In order to compute the total area "under the curve", compute $\int_0^1 (2x^2+x-3)dx + \int_1^2 (2x^2+x-3)dx$. Using the result from 1e, we can write:

$$\begin{aligned}\int_0^1 (2x^2 + x - 3)dx + \int_1^2 (2x^2 + x - 3)dx =& \left|\frac{2}{3} + \frac{1}{2} - 3 - (0)\right| + \\ & \left|\frac{2}{3}2^3 + \frac{1}{2}2^2 - 3 \cdot 2 - (\frac{2}{3} + \frac{1}{2} - 3)\right| \\ =& \frac{11}{6} + \frac{4}{3} + \frac{11}{6} \\ =& 5\end{aligned}$$