University of Mannheim
School of Social Sciences
Mathematics for Political Scientists, Fall 2022
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Solutions Set Theory II

1. Let $S = \{\text{Homer, Marge, Bart, Lisa, Maggie}\}$. Enumerate the following relations.

   (a) "is a sibling of"

   $$R = \{(B,L),(B,Ma),(L,B),(L,Ma),(Ma,B),(Ma,L)\}$$

   (M) for Marge and (Ma) for Maggie.

   (b) "is married to"

   $$R = \{(H,M),(M,H)\}$$

   (c) "is taller than"

   $$R = \{(M,H),(M,B),(M,L),(M,Ma),(H,B),(H,L),(H,Ma)\\ ,(B,L),(B,Ma),(L,Ma)\}$$

   (d) "is older than"

   $$R = \{(H,M),(H,B),(H,L),(H,Ma),(M,B),(M,L),(M,Ma)\\ ,(B,L),(B,Ma),(L,Ma)\}$$

2. Let $S = \{1,2,3,4\}$. Graph the following relations.

   (a) =

4				·
3			·	
2		·		
1	·			
=	1	2	3	4

   (b) <

4				
3				·
2			·	·
1		·	·	·
<	1	2	3	4

(c) R=$\{(1,1),(2,1),(2,2),(3,3),(4,3),(4,4)\}$

4			·	·
3			·	
2	·	·		
1	·			
R	1	2	3	4

3. Determine the following sets.

(a) The upper contour set of Lisa in 1c.

$$S = \{(B,L),(M,L),(H,L)\}$$

(b) The lower contour set of Bart in 1d.

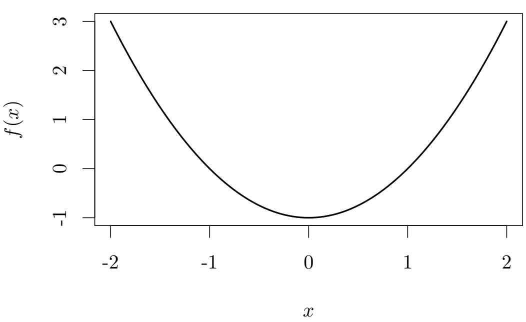
$$S = \{(M,B),(H,B)\}$$

(c) The upper contour set of 2 in 2b.

$$S = \{(1,2)\}$$

(d) Let $S = \mathbb{R}$. Determine the upper contour set of $x \in S$ given the relation $R = \{(x,y) \in S \times S | x^2 - 1 = y\}$. Graph the binary relation. Is the upper contour set convex or concave or none of both?

$f(x) = x^2 - 1$



The upper contour set is the area above the graph. It is convex.

4. Check whether the following relations are reflexive, irreflexive, transitive, complete, symmetric, antisymmetric and/or asymmetric. Also check whether they are a weak, strict, weak partial or strict partial order (or none of those).

(a) $\leq$

(b) $<$

(c) $=$

(d) $R_1 = \{(1,1),(1,2),(1,3),(1,4)\}$

(e) $R_2 =$"was born before"

(f) $R_3 = \{(a,a),(a,b)\}$

•	ref	irr	trans	com	sym	anti	asym	weak	strict	partial
$\leq$	✓	•	✓	✓	•	✓	•	✓		✓
$<$	•	✓	✓	✓	•	✓	✓		✓	✓
$=$	✓	•	✓	✓	✓	✓	•	✓		✓
R_1	✓	•	✓	•	•	✓	✓			✓
R_2	•	✓	✓	•	•	✓	•			✓
R_3	✓	•	✓	•	•	✓	•			✓

5. Let $S = \{1,2,3\}$. Show by example that...

   (a) ...if R is asymmetric, it is also antisymmetric.
   An asymmetric relation, say $>$, is:

   $$R_1 = (2,1),(3,2),(3,1)$$

   For antisymmetry is must hold that $\forall x, y, \in S, [(xRy) \wedge (yRx)] \implies (x = y)$. Since the hypothesis is always false, the statement is always true and thus, the relation is antisymmetric.

   (b) ...if R is asymmetric, it is also irreflexive.
   Consider the same relation R_1. Irreflexivity requires that $\nexists x \in S, xRx$ which is the case.

   (c) ...if R is irreflexive and transitive, it is also asymmetric. Consider the irreflexive relation

   $$R_2 = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}.$$

   To meet also transitivity, the relation reduces to

   $$R_3 = \{(1,2),(1,3),(2,3)\}.$$

   It can be seen that this relation is also asymmetric.

   (d) ...if R is symmetric and antisymmetric, it is also transitive. As an example consider the relation $=$ which is symmetric and also antisymmetric.

   $$R_4 = \{(1,1),(2,2),(3,3)\}.$$

   This relation is also trnasitive since the hypothesis is always false and therefore the statement is true.
   As another example consider the relation $\neq$. This relation is symmetric, but it is not antisymmetric and therefore not necessarily transitive.

6. Let $A = \{\{a\},\{b\},\{a,b\}\}$. Let $B = \mathcal{P}(A) \setminus \emptyset$, where $\mathcal{P}(A)$ is the *power set*, the set of all subsets of A. Define a binary relation $R \equiv \subseteq$.

(a) Explicitly enumerate B, and state its cardinality.

The power set of A minus the empty set is

$$B = \{\{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\{a\}, \{b\}\}, \{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\}\}$$

Its cardinality is 7 $(2^3 - 1)$.

(b) Prove whether R is a weak order on B.

A weak order requires reflexivity, completeness and transitivity. Consider the completeness requirement. $\{\{a\}\} \subseteq \{\{b\}\}$ is false. The reverse, $\{\{b\}\} \subseteq \{\{a\}\}$ is also false. Thus, the relation is not complete and therefore R is not a weak ordering on B.

(c) Prove whether R is a partial order on B.

A partial ordering requires reflexivity, transitivity and antisymmetry. Reflexivity requires that $\forall x$ in S, xRx. This is clearly the case, since $\subseteq$ includes equality and thus $\forall x$ in $B, x = x$. Transitivity is also given: If $x \subseteq y$ and $y \subseteq z$, then $x \subseteq z$. For antisymmetry, it needs to hold that (xRy) and (yRx) imply $x = y$. The only case for which the premise can hold for the relation $\subseteq$ is if $x = y$. Thus, $\subseteq$ is a partial order on B.

(a) Let R_1 and R_2 be transitive relations on a set S. Does it follow that $R_1 \cup R_2$ is transitive?

No. A counterexample: $S = \{1, 2\}$. $R_1 =' >' = \{(2, 1)\}$ and $R_2 =' <' = \{1, 2\}$. And $R_1 \cup R_2 = \{(1, 2), (2, 1)\}$ is not transitive.

(b) Let R_1 and R_2 be transitive relations on a set S. Does it follow that $R_1 \cap R_2$ is transitive?

Yes. Assume R_1 and R_2 are both transitive and let $(a, b), (b, c) \in R_1 \cap R_2$. Then $(a, b), (b, c) \in R_1$ and $(a, b), (b, c) \in R_2$. It is given that both R and S are transitive, so $(a, c) \in R_1$ and $(a, c) \in R_2$. Therefore $(a, c) \in R_1 \cap R_2$. This shows that for arbitrary $(a, b), (b, c) \in R_1 \cap R_2$ we have $(a, c) \in R \cap R_2$. Thus $R_1 \cap R_2$ is transitive.